Completing the Square

GCSE Maths · Worked Solutions

20 questions · 62 marks total

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Section A · Foundation

Questions 1–3 are the most straightforward; they get harder as you go on. Question 10 is the most challenging.

1 **[1 mark]**

What number must be put in the box to make $x^2 + 6x + \square$ a perfect square?

1. Halve the coefficient of x: $6 \div 2 = 3$
2. Square it: $3^2 = 9$, and indeed $x^2 + 6x + 9 = (x + 3)^2$

Answer: 9

2 **[2 marks]**

Write $x^2 + 4x + 1$ in the form $(x + a)^2 + b$

1. Half of 4 is 2, so start with $(x + 2)^2$
2. $(x + 2)^2 = x^2 + 4x + 4$, which is 4 too big.
3. So $x^2 + 4x + 1 = (x + 2)^2 - 4 + 1$

Answer: $(x + 2)^2 - 3$

3 **[2 marks]**

Write $x^2 - 6x + 10$ in the form $(x + a)^2 + b$

1. Half of −6 is −3, so start with $(x - 3)^2 = x^2 - 6x + 9$
2. That is 9 too big, so subtract 9 and add the original 10.
3. $(x - 3)^2 - 9 + 10 = (x - 3)^2 + 1$

Answer: $(x - 3)^2 + 1$

4 **[2 marks]**

Write $x^2 + 10x$ in the form $(x + a)^2 + b$

1. Half of 10 is 5, so start with $(x + 5)^2 = x^2 + 10x + 25$
2. There is no number term in the original, so subtract the whole 25.

Answer: $(x + 5)^2 - 25$

5 **[3 marks]**

(a) Write $x^2 - 8x + 20$ in the form $(x + a)^2 + b$

(b) Write down the smallest possible value of $x^2 - 8x + 20$ and the value of x that gives it.

1 **(a)** Half of −8 is −4, so $(x - 4)^2 = x^2 - 8x + 16$

2 $(x - 4)^2 - 16 + 20 = (x - 4)^2 + 4$

3 **(b)** A square is never negative, so the smallest $(x - 4)^2$ can be is 0.

4 That happens when $x = 4$, giving a smallest value of 4.

Answer: (a) $(x - 4)^2 + 4$ (b) smallest value 4, when $x = 4$

6 **[3 marks]**

Write $x^2 + 3x + 1$ in the form $(x + a)^2 + b$

1 Half of 3 is 1.5, so start with $(x + 1.5)^2$

2 $(x + 1.5)^2 = x^2 + 3x + 2.25$

3 Subtract the extra 2.25 and add the original 1: $-2.25 + 1 = -1.25$

Answer: $(x + 1.5)^2 - 1.25$

7 **[3 marks]**

Solve $x^2 + 6x + 5 = 0$ by completing the square.

1 Half of 6 is 3, so $x^2 + 6x + 5 = (x + 3)^2 - 9 + 5 = (x + 3)^2 - 4$

2 So $(x + 3)^2 - 4 = 0$, giving $(x + 3)^2 = 4$

3 Take the square root of both sides, remembering both signs: $x + 3 = \pm 2$

4 $x = -3 + 2 = -1$ or $x = -3 - 2 = -5$

Answer: $x = -1$ or $x = -5$

8 **[3 marks]**

Solve $x^2 - 4x - 1 = 0$ by completing the square, giving your answers in surd form.

1 Half of −4 is −2, so the expression is $(x - 2)^2 - 4 - 1 = (x - 2)^2 - 5$

2 $(x - 2)^2 = 5$

3 $x - 2 = \pm\sqrt{5}$

4 $x = 2 \pm \sqrt{5}$

Answer: $x = 2 + \sqrt{5}$ or $x = 2 - \sqrt{5}$

9 **[4 marks]**

A curve has equation $y = x^2 + 8x + 3$

(a) Write the equation in the form $y = (x + \text{a})^2 + \text{b}$

(b) Write down the coordinates of the minimum point.

(c) Write down the equation of the line of symmetry.

1 **(a)** Half of 8 is 4: $(x + 4)^2 = x^2 + 8x + 16$

2 $y = (x + 4)^2 - 16 + 3 = (x + 4)^2 - 13$

3 **(b)** The bracket is smallest (zero) when $x = -4$, and then $y = -13$

4 The minimum point is $(-4, -13)$.

5 **(c)** The line of symmetry passes through the minimum: $x = -4$

Answer: (a) $(x + 4)^2 - 13$ (b) $(-4, -13)$ (c) $x = -4$

10 **[4 marks]**

Solve $x^2 + 5x + 3 = 0$ by completing the square.

Give your answers to 2 decimal places.

1 Half of 5 is 2.5, so $(x + 2.5)^2 = x^2 + 5x + 6.25$

2 $x^2 + 5x + 3 = (x + 2.5)^2 - 6.25 + 3 = (x + 2.5)^2 - 3.25$

3 $(x + 2.5)^2 = 3.25$

4 $x + 2.5 = \pm\sqrt{3.25} = \pm 1.80277\ldots$

5 $x = -2.5 + 1.8028 = -0.6972\ldots$ or $x = -2.5 - 1.8028 = -4.3028\ldots$

Answer: $x = -0.70$ or $x = -4.30$ (2 d.p.)

Section B · Higher

Higher-tier style. Again, questions 1–3 are the gentlest and question 10 is the hardest.

1 **[2 marks]**

Write $x^2 - 12x + 40$ in the form $(x + \text{a})^2 + \text{b}$

1 Half of −12 is −6, so $(x - 6)^2 = x^2 - 12x + 36$

2 $-36 + 40 = 4$

Answer: $(x - 6)^2 + 4$

2 **[2 marks]**

Write down the coordinates of the minimum point of $y = (x - 3)^2 + 7$

1 The squared bracket is smallest when it equals 0, which happens at $x = 3$

2 Then $y = 0 + 7 = 7$

Answer: (3, 7)

3 **[3 marks]**

Write $2x^2 + 8x + 5$ in the form $a(x + b)^2 + c$

1. Take the 2 out of the x terms only: $2(x^2 + 4x) + 5$
2. Complete the square inside: $x^2 + 4x = (x + 2)^2 - 4$
3. $2[(x + 2)^2 - 4] + 5 = 2(x + 2)^2 - 8 + 5$

> **Answer:** $2(x + 2)^2 - 3$

4 **[3 marks]**

Write $3x^2 - 12x + 7$ in the form $a(x + b)^2 + c$

1. Take out 3 from the first two terms: $3(x^2 - 4x) + 7$
2. $x^2 - 4x = (x - 2)^2 - 4$
3. $3[(x - 2)^2 - 4] + 7 = 3(x - 2)^2 - 12 + 7$

> **Answer:** $3(x - 2)^2 - 5$

5 **[4 marks]**

(a) Write $5 + 4x - x^2$ in the form $a - (x + b)^2$

(b) Hence write down the maximum value of the expression and the value of x at which it occurs.

1. **(a)** Reorder and factor out −1 from the x terms: $-(x^2 - 4x) + 5$
2. Complete the square inside: $x^2 - 4x = (x - 2)^2 - 4$
3. $-[(x - 2)^2 - 4] + 5 = -(x - 2)^2 + 4 + 5 = 9 - (x - 2)^2$
4. **(b)** We are now **subtracting** a square, so the expression is largest when the square is 0.
5. That is when $x = 2$, giving a maximum value of 9.

> **Answer:** (a) $9 - (x - 2)^2$ (b) maximum 9 at $x = 2$

6 **[4 marks]**

Solve $2x^2 - 8x + 3 = 0$ by completing the square.

Give your answers in surd form.

1. Take out 2: $2(x^2 - 4x) + 3$
2. $x^2 - 4x = (x - 2)^2 - 4$, so the expression is $2(x - 2)^2 - 8 + 3 = 2(x - 2)^2 - 5$
3. Set to zero: $2(x - 2)^2 = 5$, so $(x - 2)^2 = \frac{5}{2}$
4. $x - 2 = \pm\sqrt{\left(\frac{5}{2}\right)} = \pm\frac{\sqrt{10}}{2}$
5. $x = 2 \pm \frac{\sqrt{10}}{2}$

> **Answer:** $x = 2 + \frac{\sqrt{10}}{2}$ or $x = 2 - \frac{\sqrt{10}}{2}$

7 **[4 marks]**

A curve has equation $y = x^2 - 6x + 11$

(a) Complete the square.

(b) Write down the turning point.

(c) Explain why the equation $x^2 - 6x + 11 = 0$ has no real solutions.

1 **(a)** Half of −6 is −3: $(x - 3)^2 - 9 + 11 = (x - 3)^2 + 2$

2 **(b)** The smallest value of the bracket is 0 at $x = 3$, giving $y = 2$

3 The turning point is a minimum at (3, 2).

4 **(c)** Setting $y = 0$ gives $(x - 3)^2 = -2$

5 A square can never be negative, so no real value of x works — the curve stays entirely above the x-axis.

Answer: (a) $(x - 3)^2 + 2$ (b) (3, 2) (c) it would need $(x - 3)^2 = -2$, which is impossible

8 **[4 marks]**

The graph of $y = x^2$ is translated to give the graph of $y = (x + 3)^2 - 5$

(a) Describe fully the transformation.

(b) Write down the turning point of the new graph.

(c) Write down the equation of the line of symmetry.

1 **(a)** The +3 is inside the bracket, so it moves the graph horizontally — and in the **opposite** direction to the sign, i.e. 3 to the left.

2 The −5 is outside, so it moves the graph 5 down.

3 The transformation is a translation by the vector 3 left and 5 down.

4 **(b)** The vertex of $y = x^2$ at (0, 0) moves to (−3, −5).

5 **(c)** The line of symmetry passes through the vertex: $x = -3$

Answer: (a) translation 3 left and 5 down (b) (−3, −5) (c) $x = -3$

9 **[4 marks]**

Find the maximum value of $7 + 6x - x^2$ and the value of x at which it occurs.

1 Factor out −1 from the x terms: $-(x^2 - 6x) + 7$

2 Complete the square inside: $x^2 - 6x = (x - 3)^2 - 9$

3 $-[(x - 3)^2 - 9] + 7 = -(x - 3)^2 + 9 + 7$

4 $= 16 - (x - 3)^2$

5 The square is smallest (zero) when $x = 3$, so the maximum value is 16.

Answer: maximum 16, when $x = 3$

10 **[5 marks]**

(a) Prove that $x^2 + 6x + 11$ is positive for every value of x.

(b) Find the least possible value of $2x^2 - 12x + 23$ and the value of x that gives it.

1 **(a)** Complete the square: half of 6 is 3, so $(x + 3)^2 = x^2 + 6x + 9$

2 $x^2 + 6x + 11 = (x + 3)^2 + 2$

3 A square is never negative, so $(x + 3)^2 \geq 0$ and the whole expression is at least 2, which is positive for every x.

4 **(b)** Take out 2: $2(x^2 - 6x) + 23$

5 $x^2 - 6x = (x - 3)^2 - 9$, so the expression is $2(x - 3)^2 - 18 + 23 = 2(x - 3)^2 + 5$

6 The least value is 5, when $x = 3$.

Answer: (a) it equals $(x + 3)^2 + 2 \geq 2$ (b) least value 5, at $x = 3$